Orthogonal and biorthogonal
texture-matched wavelet filterbanks
for hierarchical texture analysis

Th. Greiner
Schillerstr. 21
67227 Frankenthal (Pfalz)
Germany
Email: TGreiner@gmx.de
Abstract

This paper deals with the application of the wavelet transform to hierarchical texture analysis. The main goal is the design of texture-matched FIR-filterbanks as the dyadic (2-channel) orthogonal and the M-channel biorthogonal wavelet filterbanks. The new approach represents a generalization of the already known purely mathematically-oriented design methods for multirate and wavelet filterbanks. The cascaded application of the filterbanks results in a critically, by the factor $M \cdot M$ (with $M \geq 2$) downsampled, pyramidal representation of the textured image. The performance of these new filters discriminating textures is compared with the classical wavelet approaches on Brodatz’s textures.

Keywords: Hierarchical texture analysis, image pyramid, wavelet transform, multirate-filterbank, perfect reconstruction, regularity, texture-matched wavelet filterbank

Kurzfassung

Dieser Aufsatz beschäftigt sich mit der Anwendung der Wavelet-Transformation zur hierarchischen Texturanalyse. Das Ziel liegt bei dem Entwurf von texturangepaßten FIR-Filterbänken in Form der dyadischen (2-Kanal) orthogonalen Filterbank sowie der M-Kanal biorthogonalen Filterbank. Die neuen Entwurfsverfahren stellen eine Verallgemeinerung der bekannten an mathematischen Kriterien orientierten Entwurfsverfahren für Multiraten- und Wavelet-Filterbänke dar. Die kaskadierte Anwendung der Filterbänke führt zur Generierung der um den Faktor $M \cdot M$ (mit $M \geq 2$) unterabgetasteten Pyramidalendarstellung des texturierten Bildes. Die Leistungsfähigkeit der neuen Filter zur Texturunterscheidung wird auf der Basis der Brodatz Texturen mit den klassischen Wavelet-Verfahren verglichen.

Schlagwörter: Hierarchische Texturanalyse, Bildpyramide, Wavelet-Transformation, Multiraten-Filterbank, Perfekte Rekonstruktion, Regularität, Texturangepaßte Wavelet-Filterbank
Résumé:
Cet article concerne l’application de la transformation d’ondelette à l’analyse hiérarchique de texture. Le but principal est la conception de batteries de filtres RIF, adaptées à la texture telles que la batterie dyadique (2-canaux) orthogonale et la batterie biorthogonale M-canaux. La nouvelle approche représente une généralisation des méthodes de conception de batteries de filtres multi-résolution et à ondelettes, bien connues, s’appuyant sur des critères purement mathématiques. L’application en cascade des batteries de filtres conduit à la représentation pyramidale sous-échantillonnée par un facteur $M \cdot M$ (avec $M \geq 2$) de l’image texturée. La performance de ces nouveaux filtres pour la discrimination des textures est comparée avec l’approche classique par ondelettes, sur des textures de Brodatz.

Mots clés: Analyse hiérarchique de texture, représentation pyramidale, transformation d’ondelette, batterie de filtres multi-résolution, reconstruction parfaite, régularité, batterie de filtres adaptée à la texture.

Acknowledgements
This work was carried out at the University of Kaiserslautern, Department of Electrical Engineering - Control and Signal Theory, Germany. I want to thank Prof. M. Pandit for his support. Also I want to thank Prof. Y. V. Venkatesh, Indian Institute of Science, Bangalore, India, for the discussions and comments.
1 Introduction

In recent years pyramid transforms have become an important tool of digital image processing. Applications are found in image analysis and image coding. In this paper the goal of hierarchical texture analysis is the topic. For a successful texture analysis, typical texture features have to be selected and investigated for their discrimination ability. Different procedures can be developed [3-6], [11], [14], [39-44], [46-47], [50], [51], [62], [70], [82-84]. An important role is played by splitting the textured image into different feature channels [15], [44], [49], [61], [71]. This cannot sufficiently be achieved by a local support of the feature filters, because the relationships at different distances play an essential role for the complete description of the texture. Also a variation of the viewing distance influences the discrimination results. Fig. 1 shows several examples of textures [8], which demonstrate these characteristics. Therefore, an efficient algorithm has to be multichannel and multiscale [46]. This task can be carried out by a hierarchical texture analysis. But, apriori, it is not clear how many feature channels and scales are necessary for the discrimination; often an additional feature classification is required. Hence, the question arises, whether it is possible to achieve improvements by searching for characteristic features of a given texture by a texture-matched pyramidal filterbank decomposition of the image, and which discrimination results will be achieved by convolving different textures with this filterbank. The starting point will be a reference texture, which will be employed for the texture matched-filter design. Then, an optimized texture-matched filter selection should reveal significant feature differences in contrast to the reference texture "by itself", employing a smaller number of filterchannels and scales. Therefore, texture-matched filters have to be designed for multirate and wavelet analysis. As a consequence, the decomposition of the textured image in several feature sensitive pyramidal levels takes place. This in fact, is the subject of the paper.

The paper is organized as follows. The next section (Sec. 2) describes the new approaches for the design of texture-matched wavelet filterbanks. A comparison of the results with the classical wavelet approaches is described in Sec. 3. Finally, Sec. 4 summarizes the essential contents of the paper.

2. Design of texture-matched wavelet filters

2.1 Pyramid transforms

The concept of the generation of an image pyramid was introduced by Burt [9], [10], [66]. An
efficient implementation to these approaches on the basis of maximally decimated filterbanks is the wavelet transform [15], [17], [20-24], [32], [52-54], [63], [80], [88], [89]. The Wavelet-Transform is defined for quadratic integrable functions \( s(x) \in L^2(\mathbb{R}) \) and quadratic summable sequences \( s(n) \in l^2(\mathbb{R}) \) and is characterized by the scaling function \( \Phi(x) \) and the wavelet functions \( \Psi_i(x) \) with \( i=1\ldots M \) (M-number of channels). If these functions are orthogonal or at least biorthogonal an efficient signal analysis is possible. The discrete implementation of the wavelet transform takes place with a maximally decimated multirate-FIR-filterbank [19], [28], [37], [38], [57], [58], [67], [74-76], [78-80], [86] defined by the (bi-) orthogonal filters \( h_0(n) \) and \( h_i(n) \) followed by decimation (Fig. 2). An errorfree synthesis is possible by interpolation and convolving with the filters \( h'_0(n) \) and \( h'_i(n) \). A connection between the discrete analysis and synthesis filters with the continuous scaling and wavelet functions is defined by the regularity characteristic [64], [65]. The design methods can be easily extended to two-dimensional image analysis, by invoking the separability property. By cascading the 2-d-convolving and 2-d-decimating operations this leads to the pyramidal representation of the image.

A direct application of the wavelet transform for the purpose of texture analysis is possible [68]. More efficient, however, would be the selection of specific texture features, based on the characteristics of a given reference texture, in combination with the wavelet approach. Hence, the texture-sensitive features have to be taken into account during the design of the discrete scaling and wavelet filters.

For the purpose of texture analysis or in related areas of digital signal- and image processing, for example, coding, for which the consideration of known signal characteristics is an advantage, no universally valid design of signal-matched wavelets based on FIR-filters is known. Hitherto, the following results have appeared: In the work of Tewfik [69] a signal dependent wavelet transform is introduced. This approach tries, based on the design of the continuous wavelet function \( \Psi(x) \), to minimize the approximation error, caused by the restriction to a limited number of different resolution steps. Based on the difficult optimization procedure, Tewfik restricts himself to the minimization of the upper limit of the approximation error as the optimization criterion. This procedure directly influences the continuous wavelet function within the design procedure. Following the goal of a texture-matched wavelet analysis on the basis of FIR-filters, this approach does not appear to be appropriate. The analysis of periodic functions by a signal dependent wavelet design method is suggested by [60]; the design of signal dependent QMF-filters was published in [12], [25], [77], and the design of a signal dependent (not critically downsampled) pyramid is treated in [34] and [72]. The usage of signal dependent wavelet packets was proposed in [13], [87]. But design methods for signal
dependent M-channel-multirate-filters so far do not exist. Exploiting the basic relationship between
discrete wavelet transform and a multirate-system this gap will be closed in this paper.

In the following sections, the orthogonal, texture-matched two-channel and the biorthogonal,
texture-matched M-channel wavelet transform is introduced. Note, that the biorthogonal
M-channel-approach includes the biorthogonal two-channel-approach.

2.2 Main procedure

The consideration of texture-specific features by the design of one-dimensional wavelet filters
\( h_i(n) \), \( i \in \mathbb{N}_0 \), requires the introduction of quality criteria \( Q_q, q \in \mathbb{N} \). In addition, the bi-(orthogonality)
conditions \( c_k \) with \( k \in \mathbb{N}_0 \) have to be formulated. This results in an optimization problem with
constraints. For the solution of this problem, the Lagrange-approach is suggested. It is formulated in
vector notation, with the vector \( h \) representing the filter(-columns-), containing the filter coefficients,
and the Lagrange multipliers \( \lambda_k \) (vector \( \lambda \))

\[
L(h, \lambda) = \frac{1}{2} Q_q(h) - \sum_k \lambda_k \cdot c_k(h)
\]  

(1)

The optimization is performed by differentiating the Lagrangian \( L(h, \lambda) \) with respect to the vectors \( h \)
and \( \lambda \), and setting them equal to zero:

\[
\nabla_{h, \lambda} L(h, \lambda) = 0
\]

(2)

The result is a system of equations according to the separate elements of the filter vector \( h \) and the
Lagrange multipliers \( \lambda_k \). This procedure will now be explained for different design goals and quality
criteria.

2.3 Definition of the quality criteria

For the purpose of texture matching, the specification of quality criteria is required. In this
paper, the proposed quality criteria are related to the texture-matched filterbank as the Eigenfilter
according to [1]. Both the direct formulation of the quality criterion and the indirect formulation are
possible. The direct formulation employs the actual quality criterion, while the indirect formulation
at first designs a classical texture-matched filter \( h_i(n) \). This is followed by the minimization of the
mean square error between the given filter \( h_t(n) \) and the wavelet filter \( h_i(n) \) to be designed.

The direct quality criterion \( Q_1 \) is derived from the maximum-variance-approach of the discrete Karhunen-Loeve-transform:

\[
Q_1 = h_i^T \cdot C_t \cdot h_i \rightarrow \text{Max.} \tag{3}
\]

Here, \( C_t \) corresponds to the covariance matrix of the given reference texture \( t(n_1, n_2) \). The covariance matrix \( C_t \) can either be assumed for certain classes of signals with known autocorrelation function \( R_s(\Delta n_1, \Delta n_2) \) or can be calculated from the image data by averaging. For the purpose of texture analysis, the second procedure is recommended. If a sufficient texture-matched signal model is known, the parameters of this model could be used directly for texture discrimination, and a time consuming texture analysis could be avoided.

In contrast to this direct quality criterion, the indirect quality criterion \( Q_2 \) is expressed according to

\[
Q_2 = \| h_i - h_t \|^2 \rightarrow \text{Min.} \tag{4}
\]

The second quality criterion is an interesting and promising approach, in particular, if the use of already existing texture-matched (non-wavelet) filters for hierarchical texture analysis and pyramidal generation is intended. Because the realization in multirate-technique is an efficient implementation by minimizing the convolution effort, this is also an alternative for a normal filterbank, and this quality criterion should be considered.

Fundamentally, the question arises, at what resolution step the quality criteria \( Q_q \) (q=1,2) should be employed. At first, it can be based on the given resolution step \( j=0 \), so that the achieved texture-matched filter \( h_i(n) \) will be cascaded for the subsequent steps. Alternatively, for each resolution step \( j \), separate texture-matched filter \( h_i^{(j)}(n) \) will be designed using the same quality criterion \( Q_q \). Also possible is the way to employ, for each resolution step \( j \), a different quality criterion \( Q_q^{(j)} \). The basic procedure, explained here uses the first approach, and whether the other approaches are required, depends on the application.
2.4 Formulation of the constraints

In the following sections the constraints for the different filterbank approaches are discussed. These are the dyadic orthogonal case and the biorthogonal M-channel case.

2.4.1 Dyadic, orthogonal wavelet transform

While both the quality criteria are used for the orthogonal as well as for the biorthogonal wavelet transform, the constraints \( c_k \) are different. At first these constraints are explained for the orthogonal, dyadic approach using vector- and matrix notation. Here, the lowpass filter \( h_0(n) \) with filter length \( N \) will be matched to the characteristics of the reference.

The specification \( c_0 \) of the mean of the lowpass filter \( h_0(n) \) is satisfied by

\[
c_0(h_0) = h_0^T \cdot \frac{1}{h_0} - \sqrt{2}
\]

with the vector \( \frac{1}{h_0} = [1, \ldots, 1]^T \) of the length \( N \).

The demand for orthogonality is expressed by

\[
c_k(h_0) = h_0^T \cdot C_k \cdot h_0 - b_k = 1, \quad b_k = 0 \text{ for } k > 0
\]

with the matrices \( C_k \) and the constants \( b_k \).

The explicit expression of the orthogonality constraints of the filter \( h_0(n) \) depends on the filterlength \( N \). Generally, \( N_o = N/2 \) orthogonality constraints are given. The quadratic constraint-matrix \( C_k \) of the \( k \)-th constraint (\( k=1,\ldots,4 \)) with the size \( N \times N \) is symmetrical with the value 1 in the \( 2 \cdot (k-1) \)-th diagonals corresponding to the indices \( C_k[m, 2 \cdot (k-1)+m] \) and \( C_k[m, 2 \cdot (k-1)-m] \) for \( m=1,\ldots,N \) (as long as the indices lie in the interval \([1,\ldots,N]\)). All the other elements contain the value 0. Here, for the case of a filterlength of \( N=8 \), the conditions and their notations using matrices will be demonstrated. Then, the following representations are obtained:

\[
C_k = I - \text{identity matrix}
\]
\[
C_2 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\] (8)

\[
C_3 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\] (9)

\[
C_4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] (10)
The first necessary regularity condition, which requires a zero at \( z=-1 \), is already guaranteed by the mean constraint \( c_0 \) of the lowpass filter. This corresponds to the zeroth moment of \( h_1(n) \). Further zeroes may be introduced by the use of additional moment relationships of \( h_1(n) \), based on the fact that \( h_0(n) \) and \( h_1(n) \) are coupled by folding and inversion of the impulse responses:

\[
c_k(h_0) = d_k^T \cdot h_0 \quad \text{for } k > N_o
\]  

The number of degrees of freedom \( N_f \) of the Lagrange-approach must always be higher than the number of constraints, because otherwise no optimization is possible or even an overdetermined system of equations exists. The number of degrees of freedom is given for the dyadic, orthogonal approach

\[
N_f = N - N_o - N_z = N/2 - N_z
\]  

with

\begin{align*}
N & \quad \text{filter length}, \\
N_o & \quad \text{number of orthogonality constraints} = N/2 \text{ and} \\
N_z & \quad \text{number of zeroes at } z=-1,
\end{align*}

because the number of orthogonal constraints \( N_o \) depends on the filterlength \( N \), and is equal to \( N/2 \).

Due to the nonlinear coupling of the filter coefficients, which are based on the demand for translation orthogonality, a system of nonlinear constraints exists. Hence, a direct analytic solution of the Lagrange-approach is avoided, and an iterative numerical procedure is required.

### 2.4.2 Solution using the Lagrange-Newton-method

The matching of the filter coefficients described in the last sections results in a minimizing problem with nonlinear constraints. Due to the nonlinearity of the constraints, the solution of the optimizing problem is not achieved in a single step. However, an iterative procedure with the iterations steps \( p \) can be used to approximate stepwise the optimal solution.

Here, the Lagrange-Newton-approach [27] is used. At first, the nonlinear constraints on the iteration steps \( p \) are approximated by a Taylor sequence around the values \( h(p) \) and \( \lambda^{(p)} \). The Taylor approximation is cancelled with the linear term. This results in
\[ \nabla L(h^{(p)} + \Delta h^{(p)}, \lambda^{(p)} + \Delta \lambda^{(p)}) = \nabla L(h^{(p)}, \lambda^{(p)}) + \nabla^2 L(h^{(p)}, \lambda^{(p)}) \left( \begin{array}{c} \Delta h \\ \Delta \lambda \end{array} \right) = 0 \, . \] (13)

In matrix notation, this is formulated according to

\[
\begin{bmatrix}
W_q^{(p)} & -A^{(p)} \\
-A^{(p)\top} & 0
\end{bmatrix}
\begin{bmatrix}
\Delta h^{(p)} \\
\Delta \lambda^{(p)}
\end{bmatrix}
= \begin{bmatrix}
-\nabla Q_q^{(p)} + A^{(p)\top} \lambda^{(p)} \\
c^{(p)}
\end{bmatrix}
\] (14)

and the vector changes \( \Delta h \) and \( \Delta \lambda \) and \( \lambda^{(p+1)} = \lambda^{(p)} + \Delta \lambda^{(p)} \) are replaced by

\[
\begin{bmatrix}
\Delta h^{(p)} \\
\lambda^{(p+1)}
\end{bmatrix}
= \begin{bmatrix}
W_q^{(p)} & -A^{(p)} \\
-A^{(p)\top} & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
-\nabla Q_q^{(p)} \\
c^{(p)}
\end{bmatrix}
\] (15)

The matrix \( W_q \) dependent on the quality criterion \( Q_q \) is the Hesse-matrix of the Lagrange-equation, and the matrix \( A \) is the Jacobi-matrix, which contains the derivatives of the constraints \( c_k \). The iterative procedure results in:

\[ h^{(p+1)} = h^{(p)} + \Delta h^{(p)} \, . \] (16)

In [36], the explicit solution of this minimizing problem for each iteration step is given by

\[ \Delta h^{(p)} = -W^{-1} A (A^T W^{-1} A)^{-1} c + W^{-1} [A (A^T W^{-1} A)^{-1} A^T W^{-1} - J] \nabla Q_q \] (17)

and

\[ \lambda^{(p+1)} = (A^T W^{-1} A)^{-1} (A^T W^{-1} \nabla Q_q - c) \] (18)

where, on each iteration step \( p \), a solution is searched, as long as the changes of the filter coefficients \( \Delta h \) are greater than a limit \( \varepsilon \). This iterative procedure requires starting values \( h^{(0)} \) and \( \lambda^{(0)} \).

For example, the filter coefficients achieved by the classical procedure can be inserted. The Lagrange-Newton approach will now be explained for the two quality criteria \( Q_1 \) and \( Q_2 \) designing a lowpass filter with one zero at \( z=-1 \) and a filter length of \( N=8 \). The Lagrange-equation is expressed for the quality criterion \( oQ_i \) as follows
\[ L_1(h, \lambda) = h^T \cdot C \cdot h - \lambda_0(h^T 1 - \sqrt{2}) - \lambda_1(h^T C_1 h - 1) - \lambda_2(h^T C_2 h) - \lambda_3(h^T C_3 h) - \lambda_4(h^T C_4 h) . \]  

and for the quality criterion \( Q_2 \):

\[ L_2(h, \lambda) = (h - h)^T \cdot (h - h) - \lambda_0(h^T 1 - \sqrt{2}) - \lambda_1(h^T C_1 h - 1) - \lambda_2(h^T C_2 h) - \lambda_3(h^T C_3 h) - \lambda_4(h^T C_4 h) . \]

The Hesse-matrices \( W_1 \) and \( W_2 \), which contain the second derivatives of the Lagrange-equations \( L_1() \) and \( L_2() \) are given by:

\[
W_1 = \nabla^2_{h,\lambda} L_1(h, \lambda) = C_{\mu} - 2 \cdot (\lambda_1 \cdot 1 - \lambda_2 C_2 - \lambda_3 C_3 - \lambda_4 C_4) \]  

\[
W_2 = \nabla^2_{h,\lambda} L_2(h, \lambda) = I - 2 \cdot (\lambda_1 \cdot 1 - \lambda_2 C_2 - \lambda_3 C_3 - \lambda_4 C_4) . \]

The Jacobi-matrix \( A \) of the constraints \( c_k \) is determined by:

\[
A = [1, \ 2 I \cdot h, \ 2 C_2 \cdot h, \ 2 C_3 \cdot h, \ 2 C_4 \cdot h] , \]  

and the gradient of the quality criteria is determined by:

\[
\nabla h Q_1(h) = \frac{1}{2} C_{\mu} \cdot h . \]

\[
\nabla h Q_2(h) = \frac{1}{2} (h - h) . \]

2.4.3 Examples

We now present four examples of a dyadic orthogonal filterbank as designed by the explained procedure. The filter length is \( N=14 \). The regularity index \( r_8 \) was estimated after 8 iterations.
corresponding to Rioul [65]. The magnitude of the filter pair $|H_0(\omega)|, |H_1(\omega)|$, and the scaling function $\Phi(x)$ are depicted in Figs. 3-6. The filters were matched to the characteristics of the texture 'Expanded Mica'. The first 3 examples are based on the quality criterion $Q_2$, with the eigenfilters according to [1] as the given texture-matched filters. Here, the regularity is influenced by varying the number of zeroes at (z=-1). The last example is obtained for quality criterion $Q_1$. The influence of the regularity can be noticed. By increasing the regularity, the scaling function becomes smoother but keeps its basic shape (the scaling functions are determined by iteration [65]). Especially, the first example with a regularity coefficient smaller than 1, i. e., with no continuous derivative of the scaling function, shows multiple irregular peaks. These vanish with increasing regularity. The scaling function, which is achieved with the direct quality criterion, shows a similar shape. It is also evident that the magnitude response approaches, with increasing regularity, the classical shape. While in the first example one distinctive side-peak is present, this disappears completely in the third example. The magnitude response of the filter bank on the basis of the direct quality criterion has a shape which lies between those of the second and third examples.

The separable two-dimensional scaling function is presented for the first example in Fig. 7a. Based on the separable functions, it is possible to determine 4 separable impulse responses. These two-dimensional magnitude responses are shown in Fig. 7b.

### 2.5 Biorthogonal M-channel wavelet transform

By dropping the strong orthogonality constraints of the single filters, and replacing them by the biorthogonal constraints of the total multirate (analysis and synthesis) system, each filter can be matched nearly independently to the reference texture.

The necessary and sufficient condition to achieve the PR- property with a biorthogonal M-channel wavelet filterbank is expressed by the polyphase-matrix $H_p$ of the filters $h_i (i=0,...,M-1)$: the determinant $\Delta_p(z)$ of the polyphase-matrix in the z-domain must be equal to a single term as $z^{-n_0}$. In the spatial domain, this corresponds to a pure delay. Following the discussions about multirate-filterbanks for the determinant $\Delta_p(z)$ of the polyphase matrix $H_p(z)$, we can get the following structure [79]:

$$\Delta_p(z) = d_0 z^{-M(M-1)/2} + d_1 z^{-M(M-1)/2-M} + ... + d_{N-M} z^{-[(MN-M(M-1)/2)} ,$$  \hspace{1cm} (26)

with the constants $d_m$, for $m=0,...,N-M$. 

10
Now, all the coefficients except one have to be set to zero. The decision, which terms are to be selected, is free. Therefore, a system of $N-M+1$ equations, which contain products of all filter coefficients $h_i(n)$, $i=0,...,M-1$ and $n=0,...,N-1$ is achieved. For its solution, also a numerical method is necessary with $M \cdot N$ unknown coefficients.

The alternative design consists in an iterative procedure, which allows a linear solution. For each iteration step $p$ ($p=0,...,M-1$), only the filter $h_p(n)$ will be optimized, and in addition $M-1-p$ filters $h_v (v=p+1,...,M-1)$ in connection with the already optimized filters $h_i(n)$ ($i=0,...,p-1$) are prescribed. After $M$ iterations, all filters are optimized (method I). The choice of the prescribed filters $h_v(n)$ is critical. A good regularity and a good splitting in the frequency domain are required. Powers of the regularity polynomial $R_0(z)=(1+z^{-1}+...+z^{-(M-1)})^{N_z}$ as the filters $H_v(z)$ are a good choice.

The regularity of the filter to be optimized can be influenced by the power $N_z$ of the regularity polynomial $R_0(z)$, which will be brought into the $z$-transform of the prescribed filters $H_p(z)$. Here, the number of filter coefficients to be optimized will be reduced.

These iterative procedures can be further simplified. Generally, the question arises whether the goal lies in a regular filterbank or not. If no regularity is desired, $M-1$ filters (achieved by classical texture matching) can be directly used, and only the $M$-th filter has to be optimized following the PR-properties of the multirate-system (method II). If the main focus is on maximum regularity, the lowpass filter $H_0(z)$ ($M$-th filter) is chosen in a way that it contains the power $N_z$ of the regularity polynomial $R_0(z)$, and will not be further optimized. The regularity coefficient thereby is $r=N_z^{-1}$. Then $M-2$ texture-matched filters are prescribed and the $M-1$-th filter is optimized in conjunction with the PR-property (method III). This results in a maximal regular analysis-filter bank. The regularity of the synthesis-filter bank is negligible for the purposes of texture recognition.

The coefficients $d_p$ of the determinant $\Delta_p(z)$ of the polyphase matrix can be expressed as a weighted sum of the coefficients $h_p(n)$ of the filter to be optimized with length $N$:

$$d_p = \sum_n a_{mn} \cdot h_p(n) \quad \text{(27)}$$

All the coefficients $d_p$ (condensed to a vector $d_p$) and the matrix $A_p$, which contains the weighting coefficients ($A_p[m,n] = a_{mn}$), are connected by the following equation:

$$d_p = A_p \cdot h_p \quad \text{(28)}$$
The matrix $\Delta_p$ is introduced, which is equal to the matrix $\Delta_p$ with the exception of the m-th line, corresponding to the admissible delay of the determinant of the polyphase matrix. Based on these notations, the Lagrange-equation of the iterative linear biorthogonal approach is expressed by:

$$L_q^{\Delta}(h, \lambda^p) = \frac{1}{2} Q_q(h) - (\lambda^p)^T \cdot [A_p \cdot h - 0]$$, \hspace{1cm} (29)

with the vector $\Delta$ containing the Lagrange- multipliers.

Then its gradient is set equal to zero:

$$\nabla_{h, \lambda} L_q^{\Delta}(h, \lambda^p) = 0$$, \hspace{1cm} (30)

Leading to the following linear system of equations for the quality criterion $Q_1$:

$$
\begin{bmatrix}
C_n & -(A_{\Delta})^T \\
A^p & 0
\end{bmatrix}
\begin{bmatrix}
h_p \\
\lambda^p
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
$$, \hspace{1cm} (31)

and for the indirect quality criterion $Q_2$,

$$
\begin{bmatrix}
I & -(A_{\Delta})^T \\
A^p & 0
\end{bmatrix}
\begin{bmatrix}
h_p \\
\lambda^p
\end{bmatrix}
= \begin{bmatrix}
h \\
0
\end{bmatrix}
$$, \hspace{1cm} (32)

2.5.1 Examples

Below, three examples of a biorthogonal 3-channel filterbank matched to the texture 'Expanded Mica' are depicted (Figs. 8-10). The first example is based on the described iterative procedure (method I). This means that all M filters are optimized. The filter length $N_v$ of the prescribed filter is 11, the length of the optimized filter $N_p$, and $N_i$ is 17. For the second and third examples a simplified procedure is chosen. The filters $H_0(z)$ and $H_1(z)$ were prescribed as texture-matched filter (again the Eigenfilter [1]), and the filter $H_2(z)$ is optimized (method II). In the last example, the filter $H_0(z)$ as a power of the regularity polynomial $(R_0(z))^5$ is selected, $H_1(z)$ as...
Eigenfilter prescribed and again H2(z) optimized (method III). The filter length N_p (for p=1) and N_i are 11. Just the two last examples show the advantages of the biorthogonal filterbank connected with the indirect quality criterion Q_2. Now it is possible to employ the already given texture-matched filters in a multirate system, and support the pyramidal decomposition. The number of necessary convolution operations is only 1/M^j for each resolution step j in comparison with the conventional procedure.

Clearly visible are the different results of the different approaches in the frequency domain. The maxima of the frequency values coincide approximately in the three examples, however the height of the maxima varies considerably. In the second and third examples, the maxima are concentrated at deeper frequencies. Both the first designs are characterized by a regularity coefficient r less than zero. Hence no continuous scaling function exists. The scaling function of the third example, which brings in powers of the regularity polynomial, is depicted. The regularity is maximal and equal to r=4. The separable two-dimensional scaling and wavelet functions for the regular example are shown (method III). Totally nine functions and magnitude responses are shown. The functions of equal order are displayed in Fig. 11a. The two dimensional magnitude responses for these methods are shown in Fig. 11b.

3. Application and results

The comparison of the performance of methods of texture discrimination is possible by filtering the Brodatz images [8], which represent a set of textures used in the literature to demonstrate the results of new approaches. On this base a comparison with the already published approaches is possible. These images have already been presented in the first section.

The Figs. 12 and 13 show the first pyramidal levels of the textures, 'Expanded Mica' and 'Grass', achieved by the convolution with the dyadic filters matched to the texture 'Expanded Mica'. The order of the filter is H_{00}, H_{01}, H_{10}, H_{11}. The different appearance of the textures is visible. While the first texture shows bigger structures, the second texture is characterized by smaller structures. Correspondingly, the results of the wavelet analysis look alike. In Fig. 14, the wavelet pyramid is depicted, achieved by means of the 3-channel texture-matched biorthogonal wavelet filterbank using design method II. The matching occurred again for texture 'Expanded Mica'. Overall, on each step, 9 individual pictures are achieved, yielding a detailed decomposition. The order of the displayed images is determined from left to right by the filter: H_{00}, H_{01}, H_{02}, H_{10}, H_{11}, H_{12}, H_{20}, H_{21}, H_{22}.  

13
3.1 Application for the purpose of texture discrimination

The texture differences will be described by means of the energy of the extracted feature channels. This can be executed with the variance $\sigma_{ii'}^2(j)$ of the achieved feature image $S_{ii'}(j)$ generated by the filters $H_{ii'}$ ($i,i'=0,1,2$). In order to distinguish the textures, the differences of the variance for each feature image $S_{ii'}(j)$ in comparison with the reference texture are summarized with a discrimination index $e^{(j)}$. Hence, the discrimination power of the matched filter approach can be expressed by the single discrimination index. High indices show significantly recognized texture differences. Therefore, a comparison of the performance of the new and the classical wavelet filters is possible. The discrimination index is defined by:

$$
\Delta e^{(j)} = \sum_i \sum_{i'} |\sigma_{A,ii'}^2(j) - \sigma_{X,ii'}^2(j)| .
$$

The textures $X = B, C, D, E, F$ are the Brodatz textures presented in the first section. The following letters are assigned:

A - Expanded Mica, B - Cork, C - Pebbles, D - Sand, E - Paper, F - Grass

Table 1 depicts the discrimination index $\Delta e$ for the different steps. Each entry is ordered for the resolution steps $j=1,2,3$. The graphical representation of the table is shown by the Figs 15-19, again depending on the resolution step $j$.

The following results can be discussed in detail. It is shown that the texture matched method at only the first resolution step allows a clear distinction to the reference texture ‘Expanded Mica’, whereas the classical approaches require more resolution levels. The best discrimination results from the first texture-matched orthogonal design and the first texture-matched biorthogonal design. The classical approach according to Daubechies shows always on the first resolution step the least distinction, but then the features of the second and third resolution step allow a discrimination. These results can be interpreted as follows: the classical approach splits successively from higher to lower frequencies. Correspondingly, texture differences will only be recognized if the frequency components of the pictures clearly vary. If similar images are characterized by bigger sizes corresponding to deeper spatial frequencies more resolution levels are required. This is shown with the textures such as paper and grass as well as pebbles and sand (if the inverted grey levels of bigger and smaller structures are neglected). The new texture matched approaches do not employ the method of
successive frequency decomposition. Here scaled versions of reference features are selected. As a consequence, often on the first step itself features are employed, which allow a good discrimination. The next step submits a refinement by the corresponding bigger scaled version. This explains the clearer texture discrimination of the reference texture ‘Expanded Mica’. In general, it can be said, that the texture matched multiscale approaches in comparison with the classical approaches allow a more efficient and superior texture discrimination.

Furthermore, the question arises whether it will become possible to use the quantified differences to the texture ‘Expanded Mica’ on the basis of the discrimination index for the purpose of classification. The results are shown in Figs. 20-26. These are the two dimensional feature domains stretched over the resolution levels j=1 and j=2, or j=2 and j=3. Here corresponding results are achieved from the single consideration of the differences. By means of the wavelet transform according to Daubechies a classification requires the second and third resolution step. The best results are derived already on the first steps by the first orthogonal and biorthogonal texture-matched designs. The feature domain is clearly separated. The second texture-matched orthogonal approach with higher regularity shows similar results like the approach of Daubechies, but is superior to it.

Also the results demonstrate that an increasing regularity of the filters leads to worse texture discrimination. There are two reasons for this: (1) The degrees of freedom are reduced by additional regularity constraints and a fine texture discrimination is prevented. (2) The demand for regularity introduces a smoothness of the impulse responses. Hence, small texture differences cannot be recognized.

4. Conclusions

This paper has investigated the question of how an efficient texture analysis is possible by decomposing a textured image into different resolution levels. As a solution to this task, a hierarchical texture analysis based on a pyramidal decomposition of the image is given. And the wavelet transform as a solution for this task is proposed. Since the implementation of the discrete wavelet transform corresponds to a maximally decimated separable FIR-filterbank (multirate system), a systematic and efficient image analysis is achieved. The design of a texture-matched wavelet filterbank based on a solution using Lagrange-multipliers is presented by considering the multirate- and wavelet-constraints. Several quality criteria and the explicit formulation of the constraints for the different basic approaches are discussed. Finally, the performance of the new texture matched wavelet filterbanks in comparison with the classical approaches is illustrated. It is found that the new approaches can detect and classify texture differences employing a smaller number of scaled features.
5. Literature


[48] J. Kovacevic, M. Vetterli, Non-separable Multidimensional Perfect Reconstruction Filter Banks and Wavelet Bases for $\mathbb{R}^n$, IEEE-IT, 1, 1992


[66] A. Rosenfeld, Multiresolution Image Processing and Analysis, Springer 1984


[68] H. G. Stark, Multiscale Analysis, Wavelets and Texture Quality, Berichte aus der Arbeitsgruppe Technomathematik, No. 41, University of Kaiserslautern, Department of Mathematics, January 1990


[77] L. Vandendorpe, CQF Filter Banks Matched to Signal Statistics, Signal Processing No. 10, 1992,


[80] M. Vetterli, A theory of multirate filter banks, IEEE Transactions on Acoustics, Speech and


[85] M. V. Wickershauser, Acoustic Signal Compression with Wave Packets, Preprint, Yale University, 1989


